# Quick Review of Sets, Functions, Quantifiers, and Logic

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## Sets

A **set** is an unordered collection of objects. Let $X$ represent a set of numbers.

$$X = \{1, 2, 3, 5, 8\}$$

Each element in a set is assumed to be distinct. The number of distinct elements in a set is called the **cardinality** of the set, denoted as $|X|$. For example, the the cardinality of $X$ is $|X| = 5$. If you have repeated elements in a set, each element is counted only once. For example, consider the following and observe how duplicate elements do not change the set:

$$X = \{1, 2, 3, 5, 8\} = \{1, 1, 1, 2, 2, 2, 3, 5, 5, 8\}$$
$$|X| = |\{1, 2, 3, 5, 8\}| = |\{1, 1, 1, 2, 2, 2, 3, 5, 5, 8\}| = 5$$

Sets can be *finite*, containing a fixed number of elements, or *infinite*. The following are useful sets to know:

| **Symbol** | **Set** | **Example Members** |
|---|---|---|
| $\mathbb{N}$ | Natural Numbers | $1, 2, 3, 4, \cdots$ |
| $\mathbb{Z}$ | Integers | $\cdots, -2, -1, 0, 1, 2, \cdots$ |
| $\mathbb{R}$ | Real Numbers | $-2.92, 0.731, 1, \pi$ |

### Set-builder Notation

**Set-builder notation** is a mathematical notation that we often use to describe a set. We enumerate each of the elements or state the exact properties members in the set must utilize. For example, consider the following:

$$Y = \{x \mid x \text{ is natural number between 5 and 10, inclusively }\}$$

This implies that $Y = \{5, 6, 7, 8, 9, 10\}$. Suppose that we wanted to create a new set $Z$ that includes all even elements of $Y$:

$$Z = \{x \mid x \in Y \text{ and } x \bmod 2 = 0\}$$

The notation $x \in Y$ represents a locally-scoped variable $x$ for a member that is in the set $Y$ ($x = 5$, $x = 6$, $x = 7$, ...). This would be similar to `x` in the for-each loop syntax `for (int x : A)`.

## Functions

Let function $f$ be defined with the sets $X$ and $Y$, where $X$ is the set corresponding to the domain and $Y$ is the set corresponding to the codomain:

$$f : X \to Y$$

The codomain is the set of possible values that a function $f$ may output. The actual values that a set outputs is referred to as the range. For example, consider the function $g$ as defined below:

$$g : \mathbb{N} \to \mathbb{Z} \qquad \text{such that} \qquad g(n) = -1$$

This states the function $g$ accepts a parameter from the domain of natural numbers and outputs a result from the codomain of integers. We can see that $g(0) = -1$ for $n = 0$, $g(1) = -1$ for $n = 1$, and $g(1000) = -1$ for $n = 1000$. Since the function only outputs $-1$, the set representing the range is $\textit{range} = \{-1\}$. It is important to observe that $\textit{range} \subseteq \textit{codomain}$.

A function $f$ is **injective** (or **one-to-one**) if there exists exactly one $x \in X$ such that $f(x) = y$ where $y \in Y$. If $f(x_1) = y$ and $f(x_2) = y$ for some $x_1, x_2 \in X$, then the function $f$ would not be injective. This simply means that we can only have a single domain value that maps to a value in the range.

A function $f$ is **surjective** (or **onto**) if for every value $y \in Y$, where $Y$ is the set representing the codomain of the function, there exists $x \in X$, where $X$ is the set representing the domain, such that $f(x) = y$. Another way to view this is a function $f$ is surjective when the range is equal to the codomain of the function.

A function $f$ is **bijective** (or **correspondence**) when a function is both *injective* (*one-to-one*) and *surjective* (*onto*). Each element of $X$ is paired with exactly one element from $Y$. Similarly, each element from $Y$ is paired with exactly one element from $X$. There exists no unpaired elements.

A function $f$ that is bijective from set $X$ to set $Y$, then $\{(y, x) \mid (x, y) \in f\}$ is an *injective* (*one-to-one*) and *surjective* (*onto*) function from $Y$ to $X$ representing the **inverse function** $f^{-1}$. Thus, every bijective function will have an inverse function.

$$f(x) = y \qquad\qquad f^{-1}(y) = x$$

Consider the following properties for the bijective function $f : X \to Y$:

1. Each element of set $X$ is paired with exactly one element of set $Y$.
2. Each element of set $Y$ must be paired with at least one element of set $X$ ("onto $Y$").
3. No element of set $Y$ may be paired with more than one element of set $X$ ("one-to-one").

# Conditional Propositions

## Propositions

A statement with a Boolean outcome (true or false) is a **proposition**. These statements are either *true* or *false*. For example, consider the following statements:

$$\begin{aligned}
p &= \text{The course number for Data Structures is 2003} \\
q &= \text{UAFS is located in Fort Smith, Arkansas} \\
r &= x \geq 10 \text{ when } x = 35
\end{aligned}$$

The **conjunction** of two propositions $p$ and $q$ is the proposition:

$$p \wedge q \qquad \equiv \qquad p \text{ and } q$$

The **disjunction** of two propositions $p$ and $q$ is the proposition:

$$p \vee q \qquad \equiv \qquad p \text{ or } q$$

Let $r$ be some new proposition such that $r = p \wedge q$. Since $r$ is also a proposition, it will represent a result of either *true* or *false*. For example, if $r = p \wedge q$ and $s = p \vee q$, then we can define $t$ by combining $r$ and $s$ as follows:

$$\begin{aligned}
t &= r \wedge s \\
&= (p \wedge q) \wedge (p \vee q)
\end{aligned}$$

### Conditional

Let $p$ and $q$ be propositions. A **conditional proposition** is defined as

$$p \Rightarrow q$$

### Converse

If $p \Rightarrow q$ is a conditional proposition, the **converse** of the proposition is defined as:

$$q \Rightarrow p$$

### Contrapositive

If $p \Rightarrow q$ is a conditional proposition, the **contrapositive** (or transposition) of the proposition is defined as:

$$\neg q \Rightarrow \neg p$$

### Biconditional

Let $p$ and $q$ be propositions. A **biconditional proposition** is defined as

$$p \iff q$$

It can also be read as $p$ *if and only if* $q$, sometimes denoted as $p$ *iff* $q$.

## Quantifiers

Let $S$ be some set. Let $P(x)$ be some *propositional function* regarding variable $x$, where some statement is made about $x$ that is either true or false. Consider the following statement for $P(x)$:

$$P(x): \; x \text{ is greater than } 3$$

The set $S$ is the *domain of discourse* which specifies the allowable values of $x$ in $P(x)$. For our example, we will let $S = \{1, 2, 3, 4, 5\}$. For each value in $S$, the statement $P(x)$ will return either true or false. For example, consider the following outcomes:

$$P(1) = \mathit{false} \qquad P(2) = \mathit{false} \qquad P(3) = \mathit{false} \qquad P(4) = \mathit{true} \qquad P(5) = \mathit{true}$$

### Universal Quantifier

A universally quantified statement is a *statement* where every element in the set share some common characteristic. Let $P$ be some propositional function with a domain of discourse $S$. The following statement is a universally quantified:

$$\text{for every } x, P(x) \quad \equiv \quad \forall x \, P(x)$$

The symbol $\forall$ represent the universal quantifier. For our example, we will let our domain of discourse $A$ be the set $A = \{1, 2, 3, 4, 5\}$. We will use the proposition function $P(x)$ that returns true or false for the statement $x$ is greater than 3. Since a statement is either true or false, consider the following quantified statement and determine its outcome:

$$\forall x \in A,\ P(x)$$

This statement should return false since there exists some $x$ in $A$ where $x \leq 3$. Another way to consider the statement $\forall x \in A,\ P(x)$ is to consider the following as a larger statement involving conjunction:

$$\bigwedge_{x \in A} P(x) = P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$$

### Existential Quantifier

An existentially quantified statement is a *statement* where there exists at least one element in the set where some property holds. Let $P$ be some propositional function with a domain of discourse $S$. The following statement is a existentially quantified:

$$\text{there exists } x, P(x) \quad \equiv \quad \exists x\, P(x)$$

The symbol $\exists$ represent the universal quantifier. For our example, we will let our domain of discourse $B$ be the set $B = \{1, 2, 3, 4, 5\}$. We will use the proposition function $P(x)$ that returns true or false for the statement $x$ is greater than 3. Since a statement is either true or false, consider the following quantified statement and determine its outcome:

$$\exists x \in A,\ P(x)$$

This statement should return true since there exists at least one $x$ in $B$ where $x \leq 3$. Another way to consider the statement $\exists x \in B,\ P(x)$ is to consider the following as a larger statement involving disjunction:

$$\bigvee_{x \in A} P(x) = P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$$

## Propositional Logic

Propositions $p$ and $q$ are **logically equivalent** when the both $p$ and $q$ are either both true or both false.

### Logical Equivalence

1. $p \Rightarrow q \qquad \equiv \qquad p \wedge \neg q$
2. $p \Leftrightarrow q \qquad \equiv \qquad (p \Rightarrow q) \wedge (q \Rightarrow p)$

## De Morgan's Laws for Logic

1. $\neg(p \vee q) \iff \neg p \wedge \neg q$
2. $\neg(p \wedge q) \iff \neg p \vee \neg q$
3. $\neg(p \Rightarrow q) \iff p \wedge \neg q$
4. $\neg(\forall x\, P(x)) \iff \exists x\, \neg P(x)$
5. $\neg(\exists x\, P(x)) \iff \forall x\, \neg P(x)$